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Abstract: In this report, we experiment a variant of the two-level ideal algorithm for parametric shape optimization that was proposed in [5]. In the linear case, the method, referred to as the Z' method, employs a permutation operator to rearrange the eigenstructure in such a way that the new high-frequency modes are associated with large eigenvalues. As a result, the classical steepest-descent iteration can be viewed as a Jacobi-type smoother, and standard multilevel strategies be applied. An alternate method is also tested based on odd-even decoupling (L' method). For a linear model problem, both new methods are found efficient and superior to the original formulation, but the Z' method is more robust. Similar numerical results are obtained for a nonlinear model problem by considering the eigensystem of the Jacobian matrix.

Key-words: Shape optimization, Bézier parameterization, multilevel algorithms, discrete Fourier analysis

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Two Level Correction Algorithms for Model Problems

Résumé : Dans ce rapport, on expérimente une variante d'algorithme idéal à deux niveaux d'optimization de forme paramétrique originellement proposée dans [5]. Dans le cas linaire, la méthode, appelée méthode Z', s'appuye sur un opérateur de permutation pour réorganiser le système propre de telle sorte que les nouveaux modes de hautes fréquences soient associés aux grandes valeurs propres. En conséquence, la méthode classique du gradient est interprétée comme un lisseur de type itération de Jacobi, et les stratégies standard des multigrilles s'appliquent. Une méthode alternative s'appuyuant sur un découplage pair-impair est aussi testée (méthode L'). Pour un problème modèle linéaire, on trouve que les deux nouvelles méthodes sont supérieures à la formulation d'origine. Cependant la méthode Z' est plus robuste. Des résultats numériques semblables sont obtenus pour un problème modèle nonlinéaire en conidérant le système propre de la matrice jacobienne.

Mots-clés : Optimisation de Forme, paramétrisation de Bézier, algorithmes multiniveaux, analyse de Fourier discrète

1 Introduction

This report is part of a number of publications dedicated to the definition and analysis of multilevel algorithms applied to parametric shape optimization [2] [3] [4] [5] [9] [10].

In short, in our algorithms, we parameterize either the shape itself (mostly in one dimension), or the shape-deformation (in three dimensions) by the coordinates of a finite number of control points in a (tensorial) Bézier formula. These coordinates are used as design parameters to optimize some relevant physical criterion. A number of applications to aerodynamic design have been considered.

As a consequence of the well-known degree-elevation process, a given parametric shape (or shape deformation) can be represented equivalently by a formula of arbitrarily-higher degree. This basic observation has been used to define (firstly in [2] and extended in [4]) multilevel algorithms in which optimization iterations are devised using coarse and fine parameterizations in a way mimicking multigrid strategies [1] [6]. These are algorithms were found very efficient and better equipped against numerical stiffness compared to the standard formulation, particularly when combined with self-adaptive parameterization [4].

In [5] and [3], an analysis of the iterative process has been proposed. It is based on a shape-reconstruction one-dimensional model problem in which the criterion is the variance (squared L_2-norm) between the analytical definition of the shape current iterate and a target. In this way, the classical steepest-descent iteration is linear and can be interpreted as a form of Jacobi iteration involving a particular matrix. The analysis of the eigenstructure of this matrix has permitted to emphasize the resemblance of the eigenmodes with Fourier modes, and to isolate certain conceptual differences with the usual intuition. In particular, in this model optimization problem, the criterion is an integral, and not a difference operator as usually assumed in the modal analysis of multigrid. Consequently, highly-oscillatory modes ("high frequencies") are now associated with small eigenvalues, and inversely, large eigenvalues for which the standard iteration is efficient, are associated with smooth modes. Consequently, the basic iteration cannot be considered (in the canonical basis) as a smoother but inversely, as an anti-smoother [5] and [4]. This observation has led us in [5] to view the basic iteration this time as an actual smoother, but in a transformed basis, in which the eigenstructure is permuted in a way to make the pairing between eigenvectors and eigenvalues inverse. Defining the coarser level in this modified basis leads to a different definition of the coarse-parameterization correction. The main purpose of this report is to experiment numerically the corresponding two-level algorithm (Z' algorithm) and compare it with the original two-level algorithm (Y' meethod). Additionally a variant is also analyzed (L' algorithm). Lastly, some experiments are made with a nonlinear case, and an algorithm analogous to the FAS method.

2 Shape Reconstruction Problem

We briefly introduce the notations related to the basic model problem. The shape to be optimized is denoted γ and admits an analytical representation $y(x)$. Correspondingly, the

target shape is defined by $\bar{y}(x)$. The criterion to be minimized is the following:

$$J(\gamma) = \int_\gamma \frac{1}{2}(y(x) - \bar{y}(x))^2 dx \,, \tag{1}$$

If the shapes are given a Bézier parametric representation, the function $y(x)$ (and analogously $\bar{y}(x)$) is defined implicitly by the parameterization:

$$\begin{cases} x(t) = \displaystyle\sum_{k=0}^{n} B_n^k(t)\, x_k \\ y(t) = \displaystyle\sum_{k=0}^{n} B_n^k(t)\, y_k \end{cases} \tag{2}$$

where $B_n^k(t) = C_n^k\, t^k\, 91-t)^{n-k}$ is a Bernstein polynomial of degree n.

The coordinates of the control points are stored in two vectors:

$$X = \{\, x_k \,\} \qquad Y = \{\, y_k \,\} \tag{3}$$

The vector X is designated as the *support* of the parameterization. It is held fixed during the iteration, and occasionally adapted to improve the accuracy [4]. The vector Y is the vector of *design variables.* Thus, the criterion can be viewed as a function of a finite number of variables:

$$J(Y) = \int_\gamma \frac{1}{2}\Big(B_n(t)^T(Y - \bar{Y})\Big)^2\, nB_{n-1}(t)^T \Delta X^0 dt \,. \tag{4}$$

where

$$B_n(t)^T = \big(\, B_n^1(t),\, B_n^2(t),\, ...,\, B_n^n(t) \,\big) \tag{5}$$

and Δ is the backward-difference operator. For the uniform case, we know that $x^0(t)' = nB_{n-1}(t)^T \Delta X^0 = 1$, and the expression of the criterion simplifies to:

$$J(Y) = \int_\gamma \frac{1}{2}\Big(B_n(t)^T(Y - \bar{Y})\Big)^2 dt \,. \tag{6}$$

In this report, we consider two-level algorithms based on nested supports [4] [2] associated with a fine and a coarse supports such that Bézier curves represented on the coarse support can be represented exactly on the fine support by degree elevation. The basic iteration on the fine level (in our experiment, we choose $n = 8$) is the same for all our algorithms. It is *steepest descent* or *Jacobi iteration* applied to the stationarity equation. The different variants differ in the way the coarse-parameterization correction iteration is defined. On the fine level:

$$J'(Y) = AY - b \,, \tag{7}$$

here the matrix

$$A = \int_0^1 B_n(t) B_n(t)^T dt = \{ \frac{1}{2n+1} \frac{C_n^i C_n^j}{C_{2n}^{i+j}} \} . \tag{8}$$

and the right-hand side is the known vector

$$b = A\bar{Y} . \tag{9}$$

For all the methods on the fine level, we use the following classical steepest-decent iteration:

$$Y^{j+1} = Y^j - \rho(AY^j - b) , \tag{10}$$

here $j = 0, 1, 2, \cdots$, Y^0 is a given initial guess, and we assume at $j = K$ we obtain the values Y^K on fine level, and go down to coarse level to make some corrections so that the approximations can converge faster than the single parameterization approach.

We can decompose matrix A into

$$A = \Omega_n \Lambda_n \Omega_n^T , \tag{11}$$

In the diagonal matrix Λ_n, real positive eigenvalues are arranged in increasing order, and the matrix of eigenvectors Ω_n is orthogonal:

$$\Omega_n \Omega_n^T = \Omega_n^T \Omega_n = I . \tag{12}$$

Thus, the first column vectors of Ω_n are the most oscillatory.

2.1 First Coarse Level Correction Method

All throughout, primes $'$ are used to denote quantities associated with the coarse parameterization level.

In our experiments, we have used $n' = 4$ on the coarsest level:

$$J(Y') = \int_\gamma \frac{1}{2} \Big(B_n(t)^T (Y_k + E_{n'}^n Y' - \bar{Y}) \Big)^2 dt . \tag{13}$$

Let

$$Y = \underbrace{Y_k - \bar{Y}}_{\substack{\text{fine-level} \\ \text{deviation} \\ \text{from target}}} + \underbrace{E_{n'}^n Y'}_{\substack{\text{extrapolated} \\ \text{coarse-level} \\ \text{correction}}} \tag{14}$$

where Y_k is the end result of the previous fine-level iteration, and $E_{n'}^n$ is the rectangular matrix representing the elevation of degree from n' to n, here used as a transfer operator from coarse to fine parameterization:

$$E_{n'}^n = E_{n-1}^n \ldots E_{n'+1}^{n'+2} E_{n'}^{n'+1} \tag{15}$$

Note that $E^n_{n'}$ has $n+1$ rows and $n'+1$ columns. Specifically, in our experiments we have used:

$$E^8_4 = E^8_7\, E^7_6\, E^6_5\, E^5_4\,, \tag{16}$$

From Matrix Calculus, we know that

$$\frac{\partial J(Y')}{\partial Y'} \quad = \quad \frac{\partial Y}{\partial Y'}\frac{\partial J(Y')}{\partial Y}\,. \tag{17}$$

Therefore:

$$\frac{\partial Y}{\partial Y'} = (E^n_{n'})^T\,, \tag{18}$$

and

$$\frac{\partial J(Y')}{\partial Y} = A(Y_k + E^n_{n'}Y' - \bar{Y})\,, \tag{19}$$

Thus by eqns (17), (18) and (19), we obtain

$$\begin{aligned} \frac{\partial J(Y')}{\partial Y'} &= \frac{\partial Y}{\partial Y'}\frac{\partial J(Y')}{\partial Y} \\ &= (E^n_{n'})^T A(Y_k + E^n_{n'}Y' - \bar{Y}) = 0\,. \end{aligned} \tag{20}$$

In other words, we have the following new matrix equation:

$$A_{cy}Y' = b_{cy}\,, \tag{21}$$

here the cofficient matrix

$$A_{cy} = (E^n_{n'})^T A E^n_{n'}\,, \tag{22}$$

and the right side vector

$$b_{cy} = (E^n_{n'})^T A(-Y_k + \bar{Y}) = (E^n_{n'})^T(b - AY_k)\,. \tag{23}$$

We can solve the following iteration on the coarse level for correction by initializing $Y'_0 = 0$:

$$Y'^{j+1} = Y'^j - \rho(A_{cy}Y'^j - b_{cy})\,, \tag{24}$$

then we can update by $Y^K + E^n_{n'}Y'$ on the fine level. Iterations on the fine level (10) and corrections on the coarse level (24) complete a two-level correction-type ideal algorithm for the nonlinear model problem. However it takes many iterations (hundreds for this model problem) to achieve complete convergence. To speed up the rate of convergence, we can use

a better technique – Tchebychev iterations [6] on the fine level, i.e., it has three steps for each cycle:

$$\begin{aligned} Y^{j_1} &= Y^{j_0} - \tau_1(AY^{j_0} - b)\,, \\ Y^{j_2} &= Y^{j_1} - \tau_2(AY^{j_1} - b)\,, \\ Y^{j_3} &= Y^{j_2} - \tau_3(AY^{j_2} - b)\,, \end{aligned}$$

where τ_i $(i = 1, 2, 3)$ are given by:

$$\frac{1}{\tau_i} = \frac{b+a}{2} + \frac{b-a}{2}\, r_i \tag{25}$$

where $[a, b]$ is the targetted interval in the eigenvalue λ of A and $r_i = 0, \pm\sqrt{3}/2$ a root of the Tchebychev polynomial of degree 3. When we can solve coarse corrections exactly on the coarse level, we can combine the Tchebychev iterations (25) with coarse corrections on the coarse level in the matrix form [6]:

$$G_y = G_h(I - E^n_{n'}((E^n_{n'})^T A E^n_{n'})^{-1}(E^n_{n'})^T A)G_h\,, \tag{26}$$

where $G_h = (I - \tau_3 A)(I - \tau_2 A)(I - \tau_1 A)$. In other words, the complete cycle in the form of matrix G is given by

$$Y^{j+1}_g = G_y Y^j_g + b_{gy}\,, \tag{27}$$

here

$$b_{gy} = G_h(b_h - E^n_{n'}((E^n_{n'})^T A E^n_{n'})^{-1}(E^n_{n'})^T(Ab_h - b)) + b_h\,, \tag{28}$$

and

$$b_h = ((I - \tau_3 A)(I - \tau_2 A)\tau_1 + (I - \tau_3 A)\tau_2 + \tau_3 I)b\,, \tag{29}$$

note that the original matrix problem on fine level is given by $AY = b$.

2.2 Second Coarse Level Correction Method

For the coarse level correction of Z' method, we use the following scheme:

$$J(Z') = \int_\gamma \frac{1}{2}(B_n(t)^T(Y_k + Q_0 E^n_{n'} Z' - \bar{Y}))^2 dt\,, \tag{30}$$

where $Q_0 = \Omega_n P_n \Omega_n^T$, and the matrix P_n is the permutation matrix:

$$P_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}_{(n+1)\times(n+1)}.$$

The idea of the Z' method is to reverse the pairing between eigenvalues and eigenvectors by multiplying the matrix Q_0 so that larger eigenvalues pair with higher frequency on the coarse level and relaxations can remove high frequency errors efficiently [5], while Y' method does not.

Let

$$Z = Y_k - \bar{Y} + \underbrace{\left(Q_0 E^n_{n'}\right) Z'}_{\substack{\text{extrapolated} \\ \text{\& preconditioned} \\ \text{coarse-level correction}}} \tag{31}$$

By Matrix Calculus, we know that

$$\frac{\partial J(Z')}{\partial Z'} = \frac{\partial Z}{\partial Z'} \frac{\partial J(Z')}{\partial Z} . \tag{32}$$

Therefore:

$$\frac{\partial Z}{\partial Z'} = (Q_0 E^n_{n'})^T = (E^n_{n'})^T Q_0 , \tag{33}$$

and

$$\frac{\partial J(Z')}{\partial Z} = A(Y_k + Q_0 E^n_{n'} Y' - \bar{Y}) . \tag{34}$$

Thus by eqns (32), (33) and (34), we obtain

$$\begin{aligned} \frac{\partial J(Z')}{\partial Z'} &= \frac{\partial Z}{\partial Z'} \frac{\partial J(Z')}{\partial Z} \\ &= (E^n_{n'})^T Q_0 A(Y_k + Q_0 E^n_{n'} Z' - \bar{Y}) = 0 . \end{aligned} \tag{35}$$

In summary, we have the following new matrix equation:

$$A_{cz} Z' = b_{cz} , \tag{36}$$

here the matrix

$$A_{cz} = (E^n_{n'})^T Q_0 A Q_0 E^n_{n'} = (E^n_{n'})^T A_1 E^n_{n'} , \tag{37}$$

and the vector

$$b_{cz} = (E^n_{n'})^T Q_0 (b - A Y_k) , \tag{38}$$

note that matrix $A_1 = Q_0 \, A \, Q_0$, and vector $b - A \, Y_k$ is a residual on the fine level. In effect, we have preconditioned the residual $b - AY_k$ by Q_0. We found that matices A_{cy} and A_{cz} have different eigenvalues, but the same eigenvectors.

Then we can solve the following iteration on the coarse level for correction by initializing $Z'_0 = 0$:

$$Z'^{j+1} = Z'^{j} - \rho(A_{cz}Z'^{j} - b_{cz})\,. \tag{39}$$

Finally we can update by $Y^K + Q_0 E^n_{n'} Z'$ on the fine level. We can also use Tchebychev iterations on the fine level.

In our experiments, we take

$$\bar{Y}(x) = \frac{1}{2}\sin(\pi x) + \frac{1}{2}\sin(\pi x) + \cdots + \frac{1}{n^2}\sin(n\pi x) \tag{40}$$

so that all frequency modes are present in the solution, which makes computational results and our analysis more general. When we analyze our numerical experiments, we define the "error" (or iterative error) to be the difference between the current estimate Y^j obtained by different methods and the true solution $\bar{Y}$ of the discrete problem.

We use Tchebychev iterations on the fine level and solve analytically on the coarse level, i.e., use the G cycle, then we plot errors vs the number of iterations in Fig. 1, and frequency vs the number of iterations in Fig. 2. In Fig. 1, (a) shows initial errors randomly generated, (b), (c) and (d) show errors obtained by Y' and Z' methods after one, six and eleven G cycles respectively. From them, we can easily see that the results of the Z' method are superior. Now let us look these results in the frequency space, Fig. 2 clearly explains why Z' method is superior than Y' method. After one G cycle (Fig. 2 (b)), we can see that Y' method is more efficient for reducing high frequency modes on the fine level, while the Z' method is acceptable for reducing the high frequency modes on the fine level but also the low frequency modes. After six or eleven G cycles, both frequencies are greatly reduced by Z' method, while only the high frequency content is reduced by Y' method. Of course, we can also use ideas of Full Multi-grid Method (FMG) instead of just two levels, and on the coarsest level, we can use Newton iterations to solve it instead of using the direct solver. And we consider these options in a subsequent section dealing with a nonlinear problem.

2.3 Third Coarse Level Correction Method

An alternate method is devised by considering the matrix Δ defined by:

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & (-1)^{N+1} & 0 \\ 0 & \cdots & 0 & 0 & (-1)^{N+2} \end{pmatrix}_{(N+1)\times(N+1)} ,$$

The L' and Z' Methods are based on similar principles to reverse the pairing between eigenvalues and eigenvectors by a simple matrix multiply by Δ instead of computing the

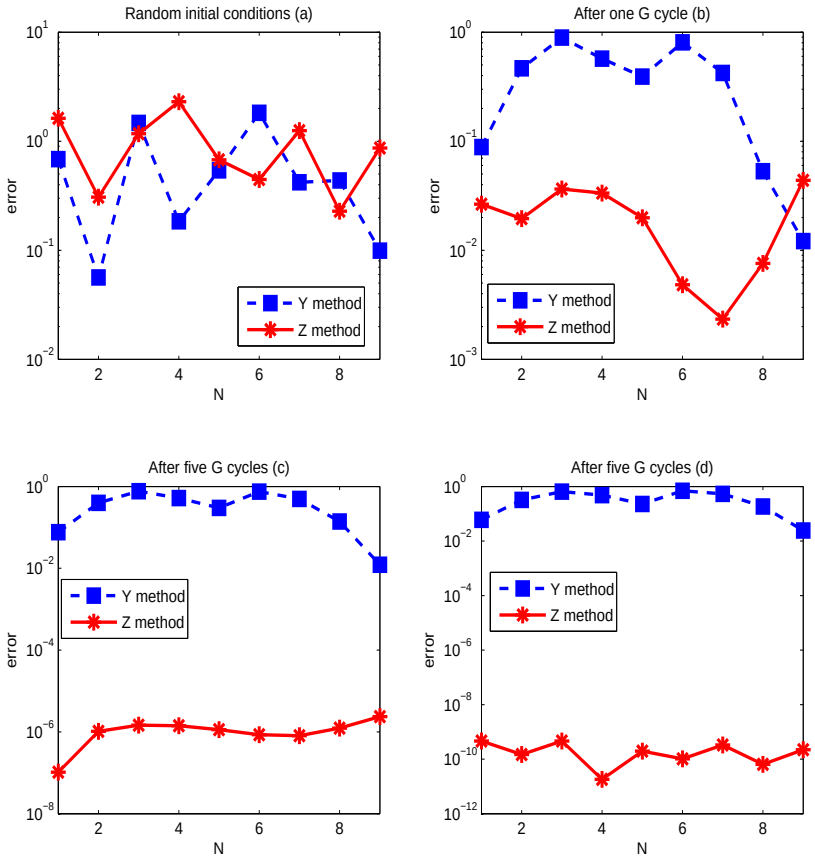

Figure 1: Nodal components. (a) shows initial errors randomly generated, (b), (c) and (d) show errors obtained by Y' and Z' methods after one, six and eleven G cycles respectively.

complicated matrix Q_0. Thus the two methods are almost same except replacing matrix Q_0 by matrix Δ in the equations, i.e., the L' method of coarse level corrections is defined by:

$$Y = Y^k + \Delta E^n_{n'} L' \,, \tag{41}$$

here Y^k is the value obtained on the fine level, L' is the coarse level correction we are looking for.

Let

$$L = Y_k - \bar{Y} + \left(\Delta E^n_{n'}\right) L' \tag{42}$$

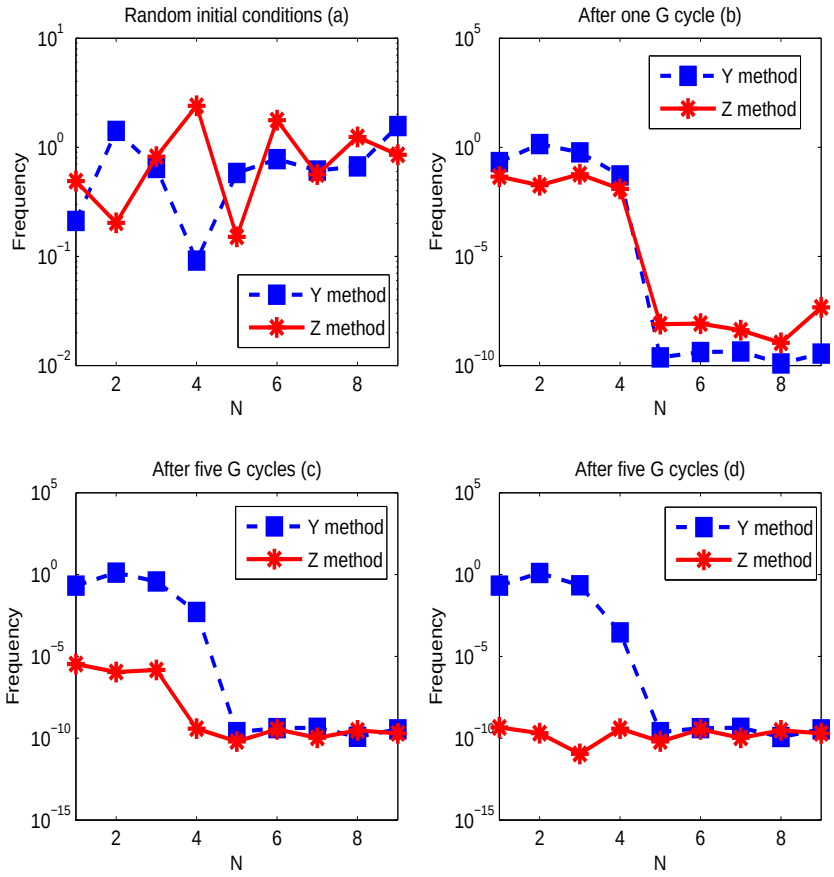

Figure 2: Frequency components. (a) shows initial frequency randomly generated, (b), (c) and (d) show frequency obtained by Y' and Z' methods after one, six and eleven G cycles respectively.

It is easy to get:

$$A_{cl} L' = b_{cl} \,, \tag{43}$$

where the matrix

$$A_{cl} = (E^n_{n'})^T \Delta A \Delta E^n_{n'} = (E^n_{n'})^T \Delta A \Delta E^n_{n'} \,, \tag{44}$$

and the vector

$$b_{cl} = (E^n_{n'})^T \Delta (b - AY_k) \,, \tag{45}$$

note that the vector $b - AY_k$ is a residual for the fine level.

Then we can solve the following iteration on the coarse level for correction by initializing $L'_0 = 0$:

$$L'^{j+1} = L'^j - \rho(A_{cl} L'^j - b_{cl}) \,. \tag{46}$$

Finally we can update by $Y^K + \Delta E^n_{n'} L'$ on the fine level. To speed up the rate of convergence, we can use Tchebychev iterations [6] on the fine level.

We use Tchebychev iterations on the fine level and solve analytically on the coarse level, i.e., use the G cycle, then we plot errors vs the number of iterations in Fig. 3, and frequency vs the number of iterations in Fig. 4. In Fig. 3, we again randomly generate initial errors, (a), (b), (c) and (d) show errors obtained by Y', L' and Z' methods after ten, fifty, one hundred and two hundred G cycles respectively. It is obvious that results obtained by Z' method is the best, now we will give some analysis for L' method. And from these graphs, we can see that before the first 50 iterations, the results of the L' method do not look better than those obtained by the Y' method, but in (c) and (d), the L' method is superior to the Y' method. From the corresponding frequency space (Fig. 4 (c) and (d)), we can see that L' method also can reduce low frequency on the fine level, however it takes more iterations to make it come, the reason is that L' method introduces some errors by the transformation matrix Δ so it requires more iterations on the fine level so that it can clear up some errors introduced by coarse corrections.

2.4 Some other experiments

In the course of our numerical experimentation on these two-level ideal algorithms for linear model problems, some additional observations have been made:

1. Some cases of non-uniform supports X have been considered for which matrix A is computed by

$$A(X)(i,j) = \frac{1}{3} \sum_{k=0}^{n-1} (x_{k+1} - x_k) \frac{C^i_n C^j_n C^k_{n-1}}{C^{i+j+k}_{3n-1}} \,, \tag{47}$$

   especially when the distribution of X is something like $(I - \cos)$ function, which has the effect of densifying points x_i close to boundaries. We found results very similar to the previous results concerning the comparison between the Y', L' and Z' methods except that sometimes results obtained by L' method seem not as robust as those obtained by the Z' method.

2. We also worked on different choices of levels of coarse corrections. In our experiments, we use the number of G cycles for various of approaches. For two level cases $n \rightleftarrows n'$,

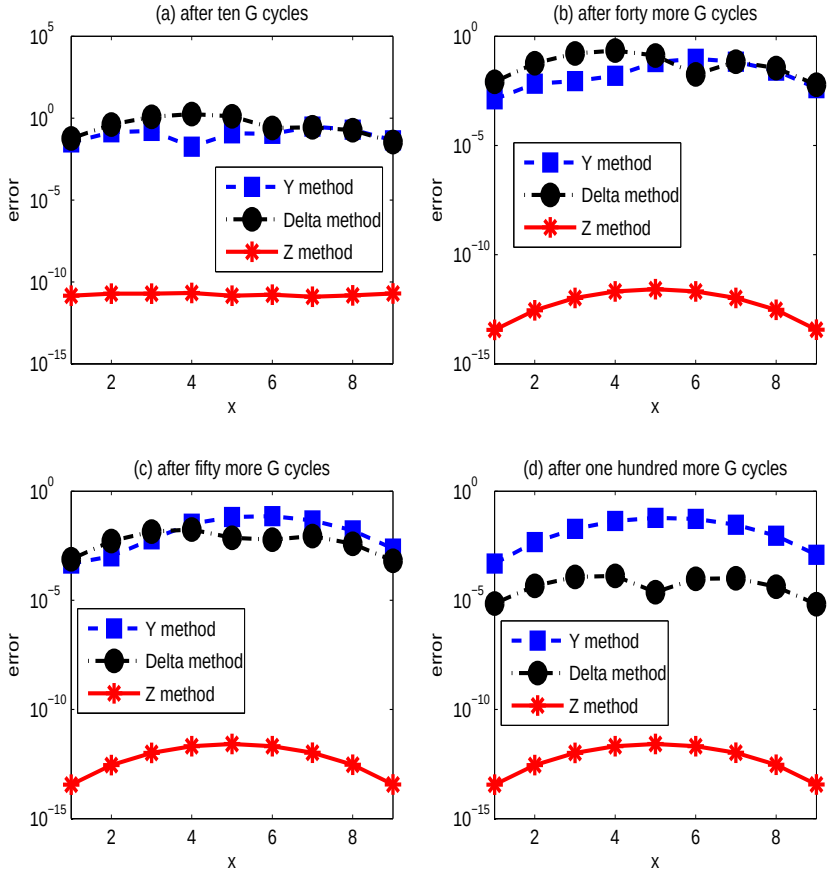

Figure 3: Nodal components. (a), (b), (c) and (d) show errors obtained by Y', L' and Z' methods after ten, fifty, one hundred and two hundred G cycles respectively.

when fine level $n = 8$, and we choose $n' = 6$, $n' = 4$ or $n' = 2$, we found that the results obtained by $n' = 6$ and $n' = 4$ are quite close, while those obtained by $n' = 2$ are not as good as the other two cases. Among the three choices, $n' = 4$ should be the best since it is good balance between the cost and accuracy.

For multi-level cases, we exploit these cases when we have the two-level case $(n = 8) \rightleftarrows (n' = 4)$ – the fine level $n = 8$ and coarse level $n' = 4$, the three-level case $(n = 8) \rightleftarrows (n' = 6) \rightleftarrows (n'' = 4)$ – the fine level $n = 8$, intermediate level $n' = 6$ and

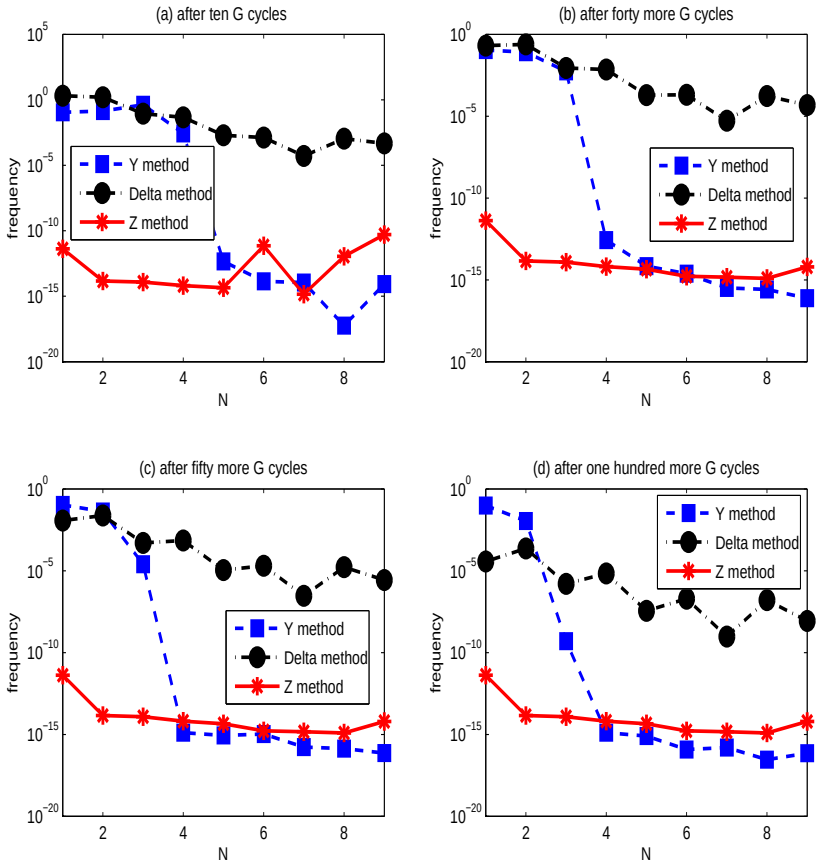

Figure 4: Frequency components. (a), (b), (c) and (d) show frequency obtained by Y', L' and Z' methods after ten, fifty, one hundred and two hundred G cycles respectively.

coarse level $n'' = 4$, and the five-level case $(n = 8) \rightleftarrows (n' = 7) \rightleftarrows (n'' = 6) \rightleftarrows (n^{(3)} = 5) \rightleftarrows (n^{(4)} = 4)$ – the finest level $n = 8$, intermediate levels $n' = 7$, $n'' = 6$, $n^{(3)} = 5$, and the coarsest level $n^{(4)} = 4$, we find the following facts:
(a) the more levels are used, the greater the accuracy (but the errors are still in the same magnitude).
(b) the more levels are used, the more regularly low frequency modes on the fine level are efficiently reduced and the more robust algorithms are.
We also find that for errors and frequency obtained by the two level case $(n = 8) \rightleftarrows$

$(n' = 7)$ are same as those obtained by the five-level case $(n = 8) \rightleftarrows (n' = 7) \rightleftarrows (n'' = 6) \rightleftarrows (n^{(3)} = 5) \rightleftarrows (n^{(4)} = 4)$. In other words, the best strategy is to use as many intermediate levels as possible.

3. On the bottom of $\mathcal{V}$ cycle, when the coarsest level n' for two level ideal algorithm or for the multi-level is small, we can use Newton iterations or solve them "exactly" for coarse corrections.

4. In the Z' method, Q_0 can be obtained in a simple way, i.e., we only need to get a similar constructed matrix which can repair eigenvalues and eigenvectors (in next section, we can use the matrix Q_0 in the linear problem for the nonlinear problem), thus we actually don't have to use Singular Value Decomposition (SVD) to compute for Q_0.

3 Nonlinear Model Problem

In this section, as we did for the linear shape reconstrunction, we extend our approaches to the nonlinear inverse-shape optimization model problem:

$$\min \mathcal{J} = \mathcal{J}(\mathrm{y(t)}) = \frac{\mathrm{p}^{\alpha}}{\mathcal{A}}\,, \tag{48}$$

in which $x(t)$ is given, smooth and monotone-increasing,

$$p = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\omega(t)\ dt\,, \qquad \mathcal{A} = \int_0^1 x'(t)y(t)\omega(t)\ dt\,, \tag{49}$$

are, for specified $\omega(t) > 0$ and $\alpha > 1$, the pseudo-length of the arc, and the pseudo-area below the arc. To make our life easier, we choose $\alpha = 2$ and $\omega(t) = 1$ for $\forall\ t$, and we know the minimum value for the nonlinear problem (48) is $\mathcal{J} = 2\pi$ (see [7] for details).

3.1 Iterations on Fine Level

Again, we assume that, for all the methods, the same classical steepest-descent iteration is applied on the fine level:

$$Y^{j+1} = Y^j - \rho\mathcal{J}'(Y^j)\,, \tag{50}$$

where $j = 0, 1, 2, \cdots$, Y^0 is a given initial guess, and we assume at $j = k$ we obtain the values Y^k on fine level, and go down to coarse level to make some corrections to accelerate the convergence in comparison with the single parameterization approach.

To best explain our ideas, we need the Jacobian matrix of $\mathcal{J}'(Y^j)$, denoted by $A_{\mathcal{J}'}$, and we will show in section 3.5 the numerical results are very good by using the Jacobian matrix $A_{\mathcal{J}'}$. Well, we must agree that in practice, more often than not, it can be difficult or

impossible to calculate formally Jacobian matrices $A_{\mathcal{J}'}$, however, but this is is not always necessary. We will explain how at the end of this section, for now, we just take for granted that we can obtain the Jacobian matrix $A_{\mathcal{J}'}$ for the model inverse shape test problem (48):

$$\mathcal{J}'(Y) = A_{\mathcal{J}'}Y - b_{\mathcal{J}'} \,. \tag{51}$$

As we did for the linear shape-reconstruction problem, we can decompose matrix $A_{\mathcal{J}'}$ into

$$A_{\mathcal{J}'} = \Omega_n \Lambda_n \Omega_n^T \,, \tag{52}$$

In the diagonal matrix Λ_n, real positive eigenvalues are arranged in increasing order, and the eigenvector matrix Ω_n is again orthogonal:

$$\Omega_n \Omega_n^T = \Omega_n^T \Omega_n = I \,. \tag{53}$$

Now we have done all the preparation, we can discuss various possible algorithms for the coarse-level correction in this nonlinear case.

3.2 First Coarse Level Correction Method

For the Y' method, we set

$$Y = Y^k + E_{n'}^n Y' \,, \tag{54}$$

where Y^K is the value obtained on the fine level, and Y' is the coarse level correction we are looking for.

By using coarse level correction (54) in the original nonlinear problem (48), we can easily obtain the following relations for the Y' correction by chain rules:

$$\mathcal{J}'(Y') = {E_{n'}^n}^T \cdot \mathcal{J}'(Y) = A'_{\mathcal{J}'}Y' - b'_{\mathcal{J}'} \,, \tag{55}$$

where $A'_{\mathcal{J}'}$ stands for the coarse level Jacobian matrix, its value can be obtained by

$$A'_{\mathcal{J}'} = {E_{n'}^n}^T A_{\mathcal{J}'} E_{n'}^n \,, \tag{56}$$

and the constant vector $b'_{\mathcal{J}'}$

$$b'_{\mathcal{J}'} = {E_{n'}^n}^T (b_{\mathcal{J}'} - A_{\mathcal{J}'}Y^K) \,, \tag{57}$$

$A_{\mathcal{J}'}$ is the Jacobian matrix on the fine level. Thus we can again employ classical steepest-descent iterations on the coarse level for correction by initializing $Y'_0 = 0$:

$$Y' \longleftarrow Y' - \rho \mathcal{J}'(Y') \,, \tag{58}$$

then we can update by $Y^K + E_{n'}^n Y'$ on the fine level. Iterations on the fine level (50) and corrections on the coarse level (58) complete a two-level correction-type ideal algorithm for

the nonlinear model problem. However it takes many iterations (hundreds for this model problem) to achieve complete convergence. To speed up the rate of convergence, we can use a better technique – Tchebychev iterations [6] on the fine level, i.e., it has three steps for each cycle:

$$\begin{aligned} Y^{j_1} &= Y^{j_0} - \tau_1 \mathcal{J}'(Y^{j_0}) \,, \\ Y^{j_2} &= Y^{j_1} - \tau_2 \mathcal{J}'(Y^{j_1}) \,, \\ Y^{j_3} &= Y^{j_2} - \tau_3 \mathcal{J}'(Y^{j_2}) \,, \end{aligned}$$

where τ_i $(i = 1, 2, 3)$ are calculated as usual in relation with the eigenvalues of the matrix $\mathcal{J}'$. We need to note that the matrix $A_{\mathcal{J}'}$ corresponds to Y^{j_0}, so we can only use Tchebychev iterations when Y^{j_0} is already close to target values otherwise the method diverges.

3.3 Second Coarse Level Correction Method

The Z' method of coarse level corrections for the nonlinear model is given as follows:

$$Y = Y^k + Q_0 E^n_{n'} Z' \,, \tag{59}$$

where Y^k is the value obtained on the fine level, Z' is the coarse level correction we are looking for, $Q_0 = \Omega_n P_n \Omega_n^T$, and P_n is the same permutation matrix as we used for linear problems. The idea of the Z' Method is to reverse the pairing between eigenvalues and eigenvectors by multiplying the matrix Q_0 so that larger eigenvalues pair with higher frequency on the coarse level and relaxations can efficiently remove the high frequency errors [5].

By using coarse level correction (59) in the original nonlinear problem (48), we can easily obtain the following relations for the Z' correction by chain rules:

$$\mathcal{J}'(Z') = {E^n_{n'}}^T \cdot Q_0 \cdot \mathcal{J}'(Y) = A'_{\mathcal{J}'} Z' - b'_{\mathcal{J}'} \,, \tag{60}$$

where $A'_{\mathcal{J}'}$ stands for the coarse level Jacobian matrix of the Z' method (we share the same notation with Y' method for simplicity), its value can be obtained by

$$A'_{\mathcal{J}'} = {E^n_{n'}}^T Q_0 A_{\mathcal{J}'} Q_0 E^n_{n'} \,, \tag{61}$$

and the constant vector $b'_{\mathcal{J}'}$

$$b'_{\mathcal{J}'} = {E^n_{n'}}^T Q_0 (b_{\mathcal{J}'} - A_{\mathcal{J}'} Y^K) \,. \tag{62}$$

Thus we can again employ the classical steepest-descent iterations on the coarse level for correction by initializing $Z'_0 = 0$:

$$Z' \longleftarrow Z' - \rho \mathcal{J}'(Z') \,, \tag{63}$$

then we can update by $Y^k + Q_0 E^n_{n'} Z'$ on the fine level. Iterations on the fine level (50) and corrections on the coarse level (63) complete another two-level correction-type ideal algorithm for the nonlinear model problem. To speed up the rate of convergence, we can use Tchebychev iterations [6] on the fine level.

3.4 Third Coarse Level Correction Method

Matrix Δ is defined as

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & (-1)^{N+1} & 0 \\ 0 & \cdots & 0 & 0 & (-1)^{N+2} \end{pmatrix}_{(N+1)\times(N+1)} ,$$

The construction is analogous to the linear case:

$$Y = Y^k + \Delta E^n_{n'} L' , \tag{64}$$

where Y^k is the vector iterate obtained on the fine level, L' is the coarse level correction we are looking for.

By using coarse level correction (59) in the original nonlinear problem (48), we can easily obtain the following relations for the L' correction by matrix calculus:

$$\mathcal{J}'(L') = {E^n_{n'}}^T \cdot \Delta \cdot \mathcal{J}'(Y) = A'_{\mathcal{J}'} L' - b'_{\mathcal{J}'} , \tag{65}$$

where $A'_{\mathcal{J}'}$ stands for the coarse level Jacobian matrix of the L' method, its value can be obtained by

$$A'_{\mathcal{J}'} = {E^n_{n'}}^T \Delta A_{\mathcal{J}'} \Delta E^n_{n'} , \tag{66}$$

and the constant vector $b'_{\mathcal{J}'}$

$$b'_{\mathcal{J}'} = {E^n_{n'}}^T \Delta (b_{\mathcal{J}'} - A_{\mathcal{J}'} Y^K) . \tag{67}$$

Thus we can again employ classical steepest-descent iterations on the coarse level for correction by initializing $L'_0 = 0$:

$$L' \longleftarrow L' - \rho \mathcal{J}'(L') , \tag{68}$$

then we can update by $Y^k + \Delta E^n_{n'} L'$ on the fine level. Iterations on the fine level (50) and corrections on the coarse level (68) complete another two-level correction-type ideal algorithm for the nonlinear model problem. To speed up the rate of convergence, we can use Tchebychev iterations [6] on the fine level.

3.5 Numerical Experiments

For the nonlinear problem (48), we take $n = 8$ for the fine level, and $n' = 4$ for the coarse level. And in all our experiments, we take the following initial guess: Then we elevate their

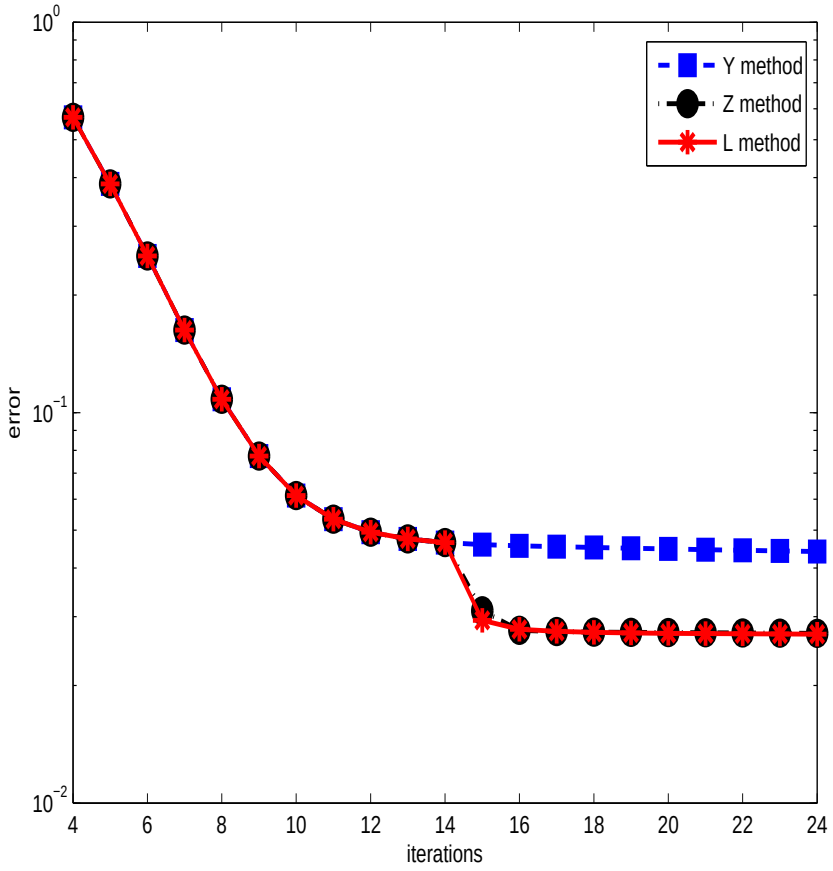

Figure 5: Comparing the results for Y', Z' and L' methods for the nonlinear model problem. First, we employ 15 iterations on the fine level ($n = 8$), and solve it completely on coarse level for both Y' and Z' methods, then go back the fine level to update values, finally do another 10 iterations. Y' method converges slowly (it still needs hundreds of iterations to fully converge), while Z' and L' methods get their full convergence for this case.

degrees by the matrix E_4^8:

$$X^0 \longleftarrow E_4^8 X^0, \qquad \text{and} \quad \mathrm{Y}^0 \longleftarrow \mathrm{E}_4^8 \mathrm{Y}^0 \,. \tag{69}$$

Table 1: Initial Values

X^0	0	0	0.077	0.409	1.0
Y^0	0	0.01	0.01	0.01	0

When we analyze our numerical experiments, we define "error" as the difference between the approximation $\mathcal{J}(Y)$ obtained by different methods concerned in our work and the true solution $\mathcal{J} = 2\pi$.

For our first numerical experiments, on the coarse level, we employ 15 iterations on the fine level, and solve it completely on coarse level for both Y' and Z' methods, then go back the fine level to update values, finally do another 10 iterations. We plot errors vs the number of iterations in Fig. 5, from it we can observe that coarse level corrections make much smaller reduction in errors of Y' method than the other two methods, and after corrections that Y' method converges slowly (it still needs hundreds of iterations to fully converge), while Z' and L' methods get their full convergence for this case. In effect of Z' and L' coarse corrections are almost same.

We define one two-level process as: first employ 35 iterations on the fine level, then do 5 iterations on the coarse level and update values on the fine level. ρ's in these iterations on both levels are taken as multiplicative inverses of maximum eigenvalues of these iteration matrices. In our second experiment, we repeat the two-level process three times and at the end of it do another 60 iterations on the fine level. We plot errors vs the number of iterations in Fig. 6. We can see that for coarse corrections of Y' method does nothing good, while coarse corrections of Z' and L' methods do excellent jobs. At the first beginning, it may appear in contradiction with the linear model problem since errors obtained by L' method are smaller than those obtained by Y' method for this nonlinear model problem. Note that for the linear case we can only obtain the exact Jacobian matrix $A_{\mathcal{J}'}$ and Q_0 for Z' method, which is not the case for nonlinear ones. Thus it is no wonder that results obtained by the L' method is the best among all considered methods for nonlinear model problems.

From the last experiment, we can see that it takes too many iterations to achieve full convergence. To speed up the rate of convergence, we use Tchebychev iterations on the fine level ($n = 8$) in our third experiment. First we employ 5 iterations (ρ's are chosen as multiplicative inverses of maximum eigenvalues of Jacobian matrices) on fine level (so that solutions are close to exact ones in some sense, which makes sure that Tchebychev iterations converge , since initial values are very bad), then we can employ 6 Tchebychev iterations (HF) on the fine level, and solve them on coarse level ($n' = 4$) "exactly" , finally go back to the fine level and do another 6 Tchebychev iterations. We put errors corresponding to the number of iterations in Fig. 7. It is obvious that Z' and L' methods converge much faster, while correction corrections of Y' method doesn't make any difference.

From what we obtain till now, it is obvious that coarse corrections of Y' method is out of the competition. In our fourth experiment, we extend Z' and L' methods to three levels

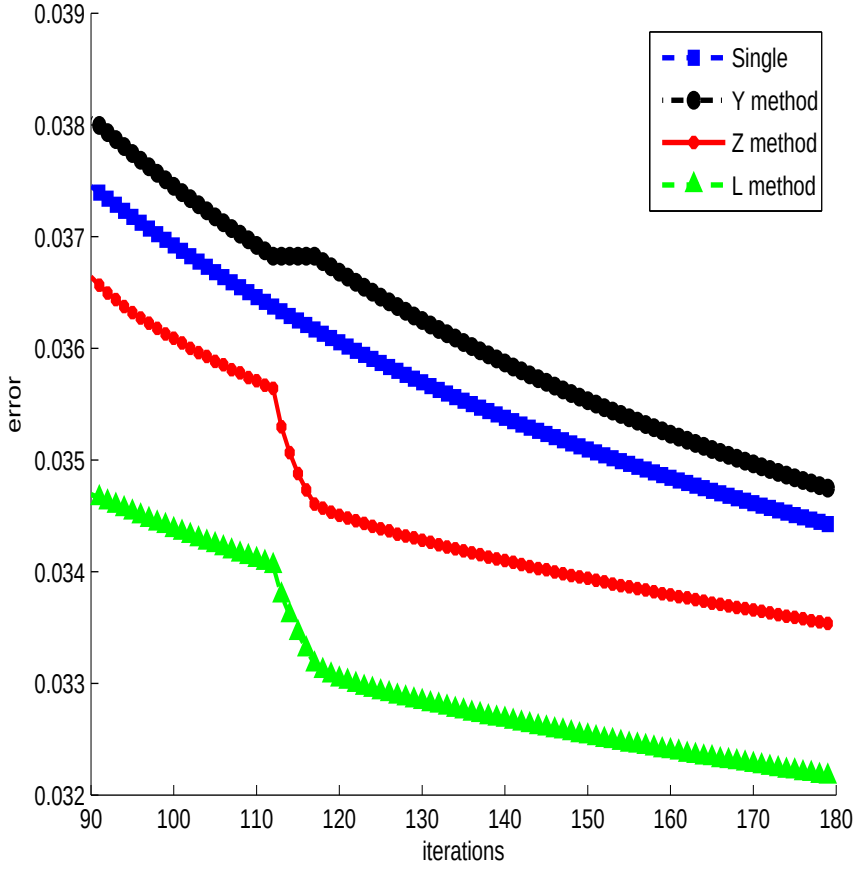

Figure 6: We define one two-level process as: first employ 35 iterations on the fine level, then do 5 iterations on the coarse level and update values on the fine level. ρ's in these iterations on both levels are taken as multiplicative inverses of maximum eigenvalues of these iteration matrices. In this numerical experience, we repeat the two-level process three times and at the end of it do another 60 iterations on the fine level. And the plot shows errors starting 90th to 180th iteration.

(fine level $n = 8$, intermediate level $n' = 4$, and coarse level $n'' = 2$) for the nonlinear model problem. For single parameterization, we just directly employ two hundred Tchebychev iterations (HF) on the fine level $n = 8$ after we use five iterations (ρ's are chosen as multiplicative inverses of maximum eigenvalues of Jacobian matrices) so that obtained

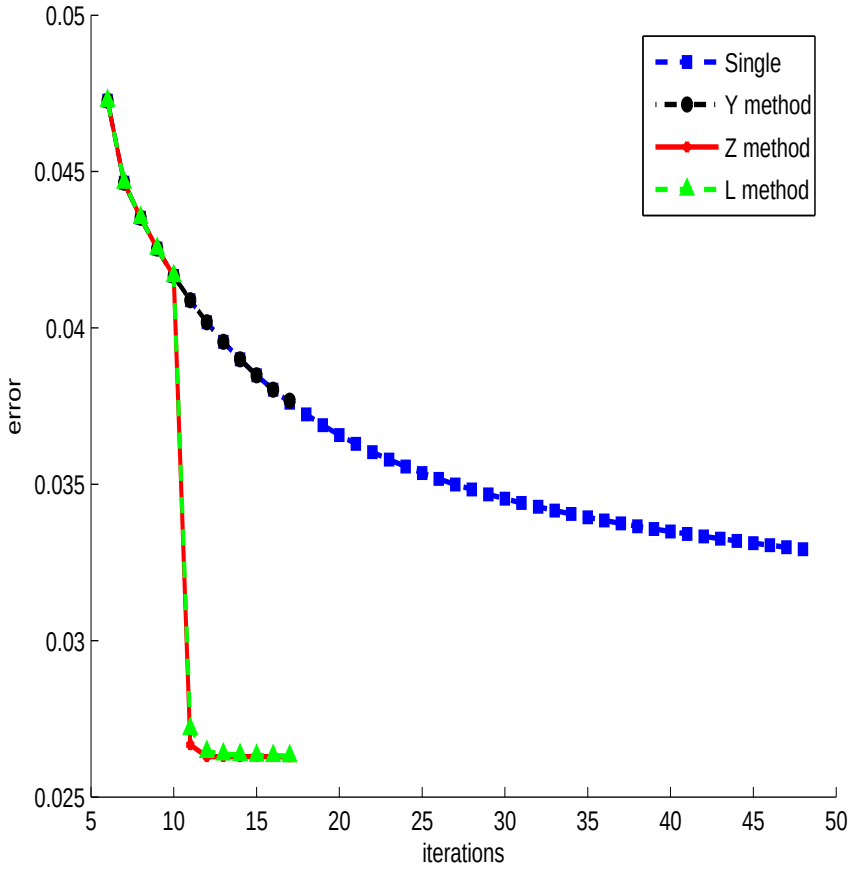

Figure 7: To speed up the rate of convergence, we use Tchebychev iterations (HF) on the fine level ($n = 8$). First we employ 5 iterations (ρ's are chosen as multiplicative inverses of maximum eigenvalues of Jacobian matrices) on fine level (so that solutions are close to exact ones in some sense, which makes sure that Tchebychev iterations converge, since initial values are very bad), then we can employ 6 Tchebychev iterations (HF) on the fine level, and solve them on coarse level ($n' = 4$) "exactly", finally go back to the fine level and do another 6 Tchebychev iterations.

approximations are close to exact ones in some sense and it makes sure that Tchebychev iterations converge (note that initial values are very bad). For the same reason, we need to do five ρ iterations, then six Tchebychev iterations (HF) on the fine level $n = 8$ and two ρ

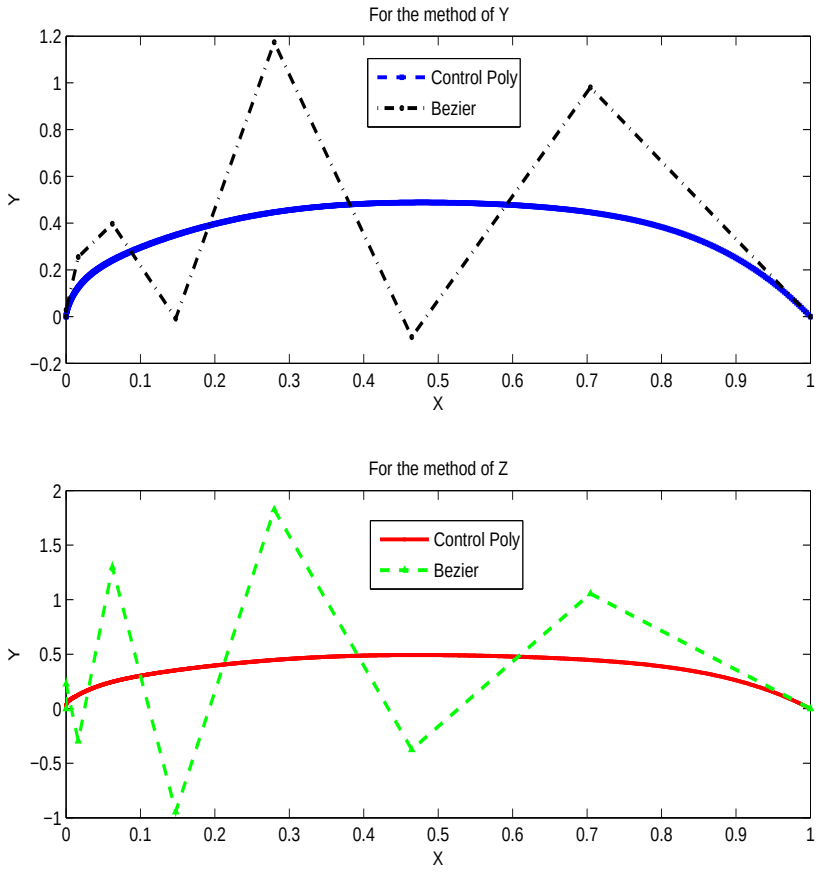

Figure 8: We plot the control polygon and Bazier profile for methods Y and Z. From it, we can easily see that the control polygon obtained by the Z method is more oscillatory, but it obtains smaller object function values and converges faster.

iterations on intermediate level $n = 4$, solve them on the coarse level $n = 2$ "exactly", after updated the correction (saw tooth algorithm), go back to fine level do another four Tchebychev iterations (HF). Last we do ten sets of two Tchebychev iterations on the fine level and intermediate level respectively, and solve corrections completely on the coarse level. Fig. 9 shows errors against the number of iterations. From the results, we can see that Z' and L' methods are excellent ways for coarse corrections.

Additionally, we have made the following interesting observations:

1. In the two-level ideal algorithms, when the fine level $n = 8$, the results are better for the coarse level $n' = 4$ than $n' = 2$, if we do the same number of iterations on the fine level and solve "exactly" on the coarse level. As far as the cost of operations is concerned, the case $n' = 2$ uses less operations on the coarse level than $n' = 4$. Thus there is not an obvious winner for this nonlinear problem.

2. In the Z' or L' three-level ideal algorithm, when the fine level $n = 8$, intermediate level $n' = 4$ and coarse level $n'' = 2$, when we already use transformation matrices Q_0 or Δ in the intermediate level to modify the pairing of eigenvalues and eigenvectors, and in the coarse level (corrections for the intermediate corrections), we do not need to employ transformation matrices Q_0 or Δ on the coarse level.

3. On the bottom of $\mathcal{V}$ cycle, when the coarsest level n' for two level ideal algorithm or $n'' = 2$ for the three level, we can also use Newton iterations to solve for coarse corrections instead of solving them "exactly".

4. In the Z' method, we can also use the Jacobian matrix of in our linear model problem to compute matrix Q_0. In fact, we found that it even converges faster when we use the Jacobian matrix of the linear model problem to compute matrix Q_0.

4 FAS

It is well known that there are two major multigrid methods [1, 8] to deal with nonlinear problems: Newton-multigrid and Full Approximation Scheme (FAS). The algorithms we used in section 3 for the nonlinear model problem are Newton's methods, in this part, we discuss the probability of applying FAS for the nonlinear model problem.

Considering the original verson of two level algorithm of FAS for solving the nonlinear problem $A(Y) = f$:

1. on the fine level n, relax the nonlinear problem $AY^j = f$ by initializing Y^0, and use Y^K as the values on the fine level after relaxations;

2. on the coarse level correction, we need to solve the following residual equation:

$$A'(Y') = A'(R_n^{n'}Y) + R_n^{n'}(f - A(Y^K)), \tag{70}$$

   note here $Y'e$ is not an error, but a coarse-level approximation, that is the reason how the name of this algorithm FAS comes from, A' stands for the nonlinear operator on the coarse level n' and $R_n^{n'}$ is used here as the restriction operator which projects from the fine level n to the coarse level n';

3. Back to the fine level n, update the values Y^K by $Y^K + E_{n'}^n(Y' - R_n^{n'}Y^K)$, here $E_{n'}^n$ is the degree elevation operator.

4.1 Restriction Operator

For the nonlinear model problem using Bézier parameterizations, we can use degree elevation matrix $E_{n'}^{n}$ as the prolongation operator in the FAS algorithm, however, as far as restriction operator $R_n^{n'}$ is concerned, it is impossible to get "exact" one. The only hope is to obtain a good approximation for the restriction operator. We can also put this question into another way, that is the get a vector Y' and it satisfies

$$E_{n'}^{n} Y' = Y \,, \tag{71}$$

where, $n > n'$, for example, $n = 8$ and $n' = 4$, so the elevation matrix $E_{n'}^{n}$ is a matrix with $(n+1)$ row and $(n'+1)$ column. It is easy to the obtain restricion operator satisfying the following

$$R_n^{n'} E_{n'}^{n} = I_{n',n'} \ , \text{ or} \qquad \mathrm{R_n^{n'} E_{n'}^{n} Y' = Y'} \,, \tag{72}$$

but it is impossible to obtain

$$E_{n'}^{n} R_n^{n'} = I_{n,n} \ , \text{ or} \qquad \mathrm{E_{n'}^{n} R_n^{n'} Y = Y} \,, \tag{73}$$

which is more related with coarse level corrections of FAS and till now how to obtain a very good approximation for it is still an open question.

It is quite natural for us to use the well developed Least Square (LS) to approximate the values for Y'

$$\min_{Y'} \| E_{n'}^{n} Y' - Y \| \,, \tag{74}$$

that is to say that the restriction operator is defined as LS. This approach yields quite good approximations for Y' on the coarse level.

In [9] [10], a new restriction operator is developed:

$$Y' = (B_{n'})^{-1} B_{n',n} Y \,, \tag{75}$$

where the invertible square matrix is given by

$$B_{n'} = \begin{pmatrix} B_{n'}^{0}(t_0) & B_{n'}^{1}(t_0) & B_{n'}^{2}(t_0) & \cdots & B_{n'}^{n'}(t_0) \\ B_{n'}^{0}(t_1) & B_{n'}^{1}(t_1) & B_{n'}^{2}(t_1) & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & B_{n'}^{n'-2}(t_{n'-1}) & B_{n'}^{n'-1}(t_{n'-1}) & B_{n'}^{n'}(t_{n'-1}) \\ & \cdots & B_{n'}^{n'-2}(t_{n'}) & B_{n'}^{n'-1}(t_{n'}) & B_{n'}^{n'}(t_{n'}) \end{pmatrix}_{(n'+1)\times(n'+1)} \,,$$

and the rectangular matrix

$$B_{n',n} = \begin{pmatrix} B_n^0(t_0) & B_n^1(t_0) & B_n^2(t_0) & \cdots & B_{n'}^n(t_0) \\ B_n^0(t_1) & B_n^1(t_1) & B_n^2(t_1) & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & B_n^{n-2}(t_{n'-1}) & B_n^{n-1}(t_{n'-1}) & B_n^n(t_{n'-1}) \\ & \cdots & B_n^{n-2}(t_{n'}) & B_n^{n-1}(t_{n'}) & B_n^n(t_{n'}) \end{pmatrix}_{(n'+1)\times(n+1)},$$

so the restriction operator is defined as $R_n^{n'} = (B_{n'})^{-1} B_{n',n}$. This method makes sure that $Y'(t_i) = Y(t_i)$ for any partition set $\{t_i\}_{i=0,1,\cdots,n'}$.

Both approaches for the restriction operator can give good approximations for Y' from our numerical experience ($n = 8, n' = 4$), the relative error between values of $\int_0^1 Y(t)dt$ on the fine level and on the coarse level are quite close (note that $Y(t) = \sum_{i=0}^{n} B_n^i(t) Y_i$ for the fine level and $Y(t) = \sum_{i=0}^{n'} B_{n'}^i(t) Y_i'$ for the coarse level) is about 0.037, while the relative error between residuals on the fine level and on the coarse level are quite large (almost 1), which contributes the failure for computing $A'(R_n^{n'} Y)$ on the coarse level introduces a comparable big error in the second step of FAS.

In the two level ideal algorithm of our nonlinear model problem, the linearized approach turns out to be identical to the algorithm of FAS when the coarse level is solved "exactly". Note in our nonlinear model problem, the coarse level matrix $A' = A'_{\mathcal{J}'}$. According to the algorithm of FAS, we can obtain:

$$\begin{aligned} Y^k &\longleftarrow Y^k + E_{n'}^n (Y' - R_n^{n'} Y^k) \\ &= Y^k + E_{n'}^n ((A'_{\mathcal{J}'})^{-1} (A'_{\mathcal{J}'} R_n^{n'} Y^k + R_n^{n'} (f - A(Y^k))) - R_n^{n'} Y^k) \\ &= Y^k + E_{n'}^n ((R_n^{n'} Y^k + (A'_{\mathcal{J}'})^{-1} R_n^{n'} (f - A(Y^k))) - R_n^{n'} Y^k) \\ &= Y^k + E_{n'}^n ((A'_{\mathcal{J}'})^{-1} R_n^{n'} (f - A(Y^k))) , \end{aligned}$$

which is the "exact" coarse level solver for Y' method we used in section 3 when we take $R_n^{n'} = (E_{n'}^n)^T$, similarly for other coarse correction methods.

5 Conclusions

In this report, two variants of the original two-level ideal algorithms have been proposed and tested on a linear and a nonlinear model problem. The definitions involve a permutation matrix and the eigenvector matrix. In the nonlinear case, this requires approximating the Jacobian matrix, and identifying at least approximately the eigenvectors. These tasks may reveal complex for more realistic models.

Nevertheless, in such approximations can be made, the present study indicates that the resulting new variants, and particularly the more robust Z' method, demonstrate significantly superior convergence properties.

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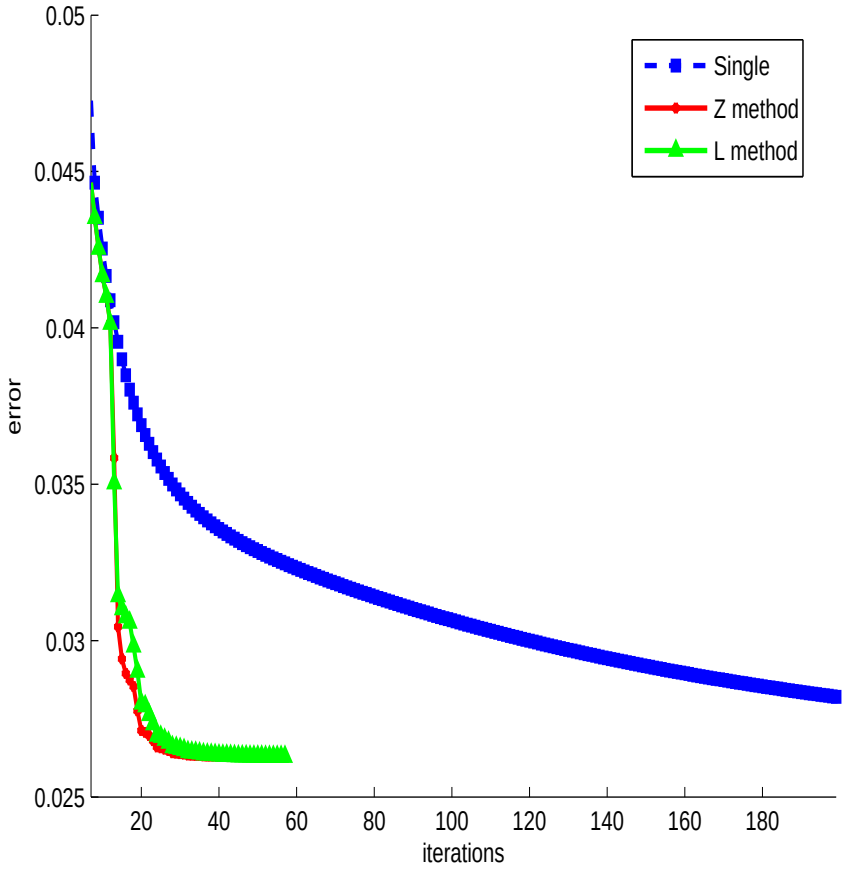

Figure 9: We extend Z' and L' methods into three levels (fine level $n = 8$, intermediate level $n' = 4$, and coarse level $n'' = 2$) for the nonlinear model problem. For single parameterization, we just directly employ two hundred Tchebychev iterations (HF) on the fine level after we use five iterations (ρ's are chosen as multiplicative inverses of maximum eigenvalues of Jacobian matrices) so that obtained approximations are close to exact ones in some sense and it makes sure that Tchebychev iterations converge (note that initial values are very bad). For the same reason, we need do five ρ iterations, then six Tchebychev iterations on the fine level and two ρ iterations on the intermediate level, solve them on the coarse level "exactly", after updated the correction (saw tooth algorithm), go back to fine level do another four Tchebychev iterations. Last we do ten sets of two Tchebychev iterations on the fine level and intermediate level respectively, and solve corrections completely on the coarse level.

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